THE

MATHEMATICAL GAZETTE.

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THE DISCUSSION OF CERTAIN POWER-SERIES.

I QUITE agree with Mr. Hardy (*Gazette*, vol. 3, p. 284) in his contention that the real basis of the power-series for $\sin x$ and $\cos x$ is given by the two equations

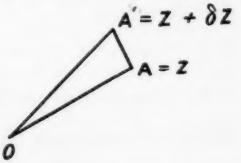
$$y = \cos x + i \sin x, \quad \frac{dy}{dx} = iy. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

But I am inclined to doubt whether at present the difficulties of exposition will not outweigh the advantages of his method (iii); and for this reason the following method may be worth considering, although it is a fairly obvious modification of my previous note (*Gazette*, vol. 3, p. 85). The only preliminary knowledge implied is that of the elementary differential calculus, and of Argand's diagram.

Lemma.

If Z is a complex variable depending in any way on a real positive increasing variable x, we have

$$\frac{d}{dx}|Z| \leqq \left|\frac{dZ}{dx}\right|.$$



For, in the diagram, we have

$$OA = |Z|, \; OA' = |Z + \delta Z|, \; AA' = |\delta Z|,$$

and so

$$\frac{d}{dx}|Z| = \lim_{\delta x = 0} \frac{OA' - OA}{\delta x} \leqq \lim_{\delta x = 0} \frac{AA'}{\delta x} \leqq \left|\frac{dZ}{dx}\right|.$$

Thus if a real positive function X, increasing with x, can be found such that $\left|\frac{dZ}{dx}\right| \leqq \frac{dX}{dx}$, we have $\frac{d}{dx}[X - |Z|] \geqq 0$, and accordingly $X - |Z| \geqq 0$, for positive values of x, provided that X and $|Z|$ are zero for $x = 0$.

If we assume that y can be expanded in a power-series, and that the series can be differentiated term-by-term, there is no difficulty in seeing that y must be equal to

$$1+ix+\frac{1}{2!}(ix)^2+\frac{1}{3!}(ix)^3+\ldots.$$

But to obtain a rigid proof of the equation we must proceed differently: consider the functions

$$\left.\begin{aligned} &Z_0=y,\ Z_1=y-1,\ Z_2=y-(1+ix), \\ &Z_n=y-\left[1+ix+\frac{1}{2!}(ix)^2+\ldots+\frac{1}{(n-1)!}(ix)^{n-1}\right] \end{aligned}\right\} \quad \ldots(2)$$

Then from (1) and (2) we see that

$$\frac{dZ_1}{dx}=iZ_0,\ \frac{dZ_2}{dx}=iZ_1,\ \ldots,\ \frac{dZ_n}{dx}=iZ_{n-1}. \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Thus from (3) we have

$$\left|\frac{dZ_1}{dx}\right| = |Z_0| = |y| = 1,$$

and therefore since $Z_1=0$ for $x=0$, we find from the lemma,

$$|Z_1| \leqq x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From (3) and (4) we get

$$\left|\frac{dZ_2}{dx}\right| = |Z_1| \leqq x,$$

and hence

$$|Z_2| \leqq \tfrac{1}{2}x^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Proceeding thus we find, step by step,

$$|Z_n| \leqq \frac{1}{n!}x^n. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)^*$$

Hence

$$\lim_{n=\infty} Z_n = 0,$$

and therefore

$$y=1+ix+\frac{1}{2!}(ix)^2+\ldots \text{ to } \infty.$$

It is easy to prove by exactly the same method that if η is a function (real) such that $\frac{d\eta}{dx}=\eta$, then

$$\left|\eta-\left(1+x+\frac{1}{2!}x^2+\ldots+\frac{1}{(n-1)!}x^{n-1}\right)\right| < \frac{1}{n!}\mathrm{H}x^n \quad \ldots\ldots(7)$$

where H is an upper limit to $|\eta|$ between 0 and x.

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* The remainder formula (6), is new as far as I know.

Although a little different from the main object of this note, it may be useful to point out the corresponding arrangements for dealing with the binomial expansion. Let us write

$$X_n = 1 + mx + m_2x^2 + \ldots + m_{n-1}x^{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where the coefficients are of the usual form.

Thus
$$(1+x)\frac{dX_n}{dx} = mX_n - (m-n+1)m_{n-1}x^{n-1}$$
$$= mX_n - n(m_nx^{n-1}), \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

or*
$$\frac{d}{dx}\left[\frac{X_n}{(1+x)^m} - 1\right] = -\frac{n(m_nx^{n-1})}{(1+x)^m}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Hence, when x is positive, (10) gives

$$\frac{d}{dx}\left|\frac{X_n}{(1+x)^m} - 1\right| \leqq n|m_n|x^{n-1}$$

so that
$$\left|\frac{X_n}{(1+x)^m} - 1\right| \leqq |m_n|x^n,$$

or
$$|X_n - (1+x)^m| \leqq |m_n| . x^n(1+x)^m. \quad \ldots\ldots\ldots\ldots(11_1)$$

But, if x is negative and greater than $-a$, where a is a positive proper fraction, we get from (10)

$$\left|\frac{X_n}{(1+x)^m} - 1\right| \leqq \frac{|m_n|a^n}{(1-a)^m},$$

or
$$|X_n - (1+x)^m| \leqq |m_n|\frac{a^n}{(1-a)^m}. \quad \ldots\ldots\ldots\ldots\ldots(11_2)$$

T. J. I'A. BROMWICH.

ON THE ADJUSTMENT OF KATER'S PENDULUM.

THE measurement of the acceleration due to gravity by means of the compound pendulum is a very common laboratory experiment; but there are certain simple principles involved in the theory of it which are not, as it seems to me, sufficiently emphasized by teachers, and which are not stated (at all events explicitly) in any text-book known to me. It may be of use to some to enunciate them here, and to shew their importance both to the instrument-maker and the student.

1. One question that sometimes arises is, What will be the effect of shifting an adjustable load to different places along the bar of a compound pendulum?

*I suppose that the rule for differentiating x^m has been proved *independently of the binomial theorem*; the proofs which depend on the binomial (for indices which are not positive integers) appear to me objectionable in every way, at least for beginners.

It is easy to predict the effect of *adding* a particle; but what will be the effect of moving a particle already present? The first case may be at once dismissed by saying that if a particle be added above the centre of oscillation, it tends to swing as a simple pendulum of shorter period, and therefore accelerates the pendulum; while if added below, it retards it. The second case requires further consideration. Let us suppose the particle to be moved along a straight line, called the axis of the pendulum, passing through the centre of suspension and the centre of gravity. Let the particle be of mass m, the mass of the rest of the pendulum being M; let the positions of m and the C.G. of M respectively be denoted by co-ordinates x, h, reckoned along the axis from the centre of suspension. Let the original and altered co-ordinates of m be x_1, x_2, respectively; let the moment of inertia of M about the knife-edge be I; let t_1, t_2 be the periods corresponding to x_1, x_2, and let l_1, l_2 be the corresponding simple equivalent pendulums. Then $t_2 \gtrless t_1$ according as $l_2 \gtrless l_1$, or as $\frac{I+mx_2^2}{Mh+mx_2} \gtrless \frac{I+mx_1^2}{Mh+mx_1}$. Now the alteration in the numerator is $m(x_2^2-x_1^2)$, and this expressed as a fraction of the original numerator is $\frac{m(x_2^2-x_1^2)}{I+mx_1^2}$; similarly the fractional increase in the denominator is $\frac{m(x_2-x_1)}{Mh+mx_1}$. The above condition amounts then to $\frac{m(x_2^2-x_1^2)}{I+mx_1^2} \gtrless \frac{m(x_2-x_1)}{Mh+mx_1}$, *i.e.* to $(x_2+x_1) \gtrless l_1$, when $x_2 > x_1$; if $x_2 < x_1$, the signs of inequality must be reversed. This condition may be put into a very simple form. Let us call the point which lies midway between the centres of suspension and oscillation for the *first* position of m the "middle point"; let us call points on the axis situated symmetrically with respect to the middle point "symmetric points." Then if a particle is initially above the middle point, raising it will retard the pendulum, lowering it to any point short of the symmetric point will accelerate it, lowering it beyond the symmetric point will retard it. If it is initially below the middle point, lowering it will increase the period, raising it to any position below the symmetric point will decrease it, raising it still further will increase it. The disadvantage of this method of expressing the results is that we utilise l_1, which is not a constant. Let us now proceed rather differently, taking L as the simple pendulum equivalent to the compound pendulum *without the particle*; thus L is constant.

We wish to discover a value x_2 which will give to the expression $\frac{I+mx^2}{Mh+mx}$ the value $\frac{I+mx_1^2}{Mh+mx_1}$. This leads to a quadratic,

of which of course one root is x_1; the other is $\frac{I-Mhx_1}{Mh+mx_1}$. Putting $L=\frac{I}{Mh}$, this reduces to $x_2=\frac{L-x_1}{1+\frac{mx_1}{Mh}}$. Moving the particle within the limits x_1 and x_2 as given by this expression will accelerate the pendulum; transferring it to a point outside these limits will retard it. Moving it from the upper limit to any position short of the lower limit increases the restoring couple more than the moment of inertia; moving it beyond the lower limit increases the moment of inertia more than the couple; raising it from the upper limit to any position short of the knife-edge decreases the couple more than the moment of inertia; raising it still higher will increase the moment of inertia while still decreasing the couple. Of course there must be positions of m which will make the period a minimum; these correspond to the cases when

$$x_2=x_1=-\frac{Mh}{m}+\sqrt{\left(\frac{Mh}{m}\right)^2+\frac{Mh}{m}.L}$$

and when

$$x_2=x_1=-\frac{Mh}{m}-\sqrt{\left(\frac{Mh}{m}\right)^2+\frac{Mh}{m}.L};$$

when m is a small fraction of M, the first will be very little below the middle point of L, while the second will be far above the knife-edge and need not be considered.

2. If we suppose next that the mass m is not a particle but has an appreciable moment of inertia about its own C.G., allowance can readily be made for this. In the first case, that of addition of load, the altered period corresponds to a simple pendulum $\frac{I+i+mx^2}{Mh+mx}$, where i denotes the moment of inertia of m about its own C.G. The question is then how this expression compares with $\frac{I}{Mh}$, *i.e.* with L; and there is no difference between them provided the C.G. of m is placed where

$$x=\frac{L}{2}\pm\frac{1}{2}\sqrt{L^2-4.\frac{i}{m}}.$$

When the radius of gyration of m about its C.G. is small compared with L, these two positions are respectively just below the knife-edge and just above the centre of oscillation of the pendulum unloaded. If m be attached at any intermediate position it decreases the period; if outside these limits it has a retarding effect.

So far as transferring m from one position to another is concerned, the only change in the previous investigation is that

$I+i$ replaces I, so that L must now be taken to mean the simple pendulum equivalent to the compound one with m so fixed that its C.G. is at the knife-edge.

3. The whole of the above discussion may be represented in a convenient graphical manner as follows. Let the axis of the pendulum be taken as x-axis, and at each point erect an ordinate equal to the length of the simple pendulum equivalent to the compound one when m is fixed at the point in question. The equation of the graph so obtained is $y=\frac{I+i+mx^2}{Mh+mx}$, which is of the form $x^2-xy=$a constant, and represents a hyperbola. One of its asymptotes is $y=x-\frac{Mh}{m}$, the other is $x=-\frac{Mh}{m}$. These lines cross the x-axis at the two points where m would have the same moment as M about the knife-edge; the latter of the two corresponds to period infinity, as is obviously right since the C.G. of the whole pendulum is at the knife-edge. If m be placed beyond the point where $x=-\frac{Mh}{m}$, we obtain the other branch of the hyperbola, of which the ordinates are negative. This of course is a sign that under the increased influence of m the pendulum is now hanging the other way up. Provided this is borne in mind, the ordinates may be plotted as if they were positive, and the second branch will then be a reflection of the first in the asymptote. The equivalent simple pendulum is a minimum for points given by

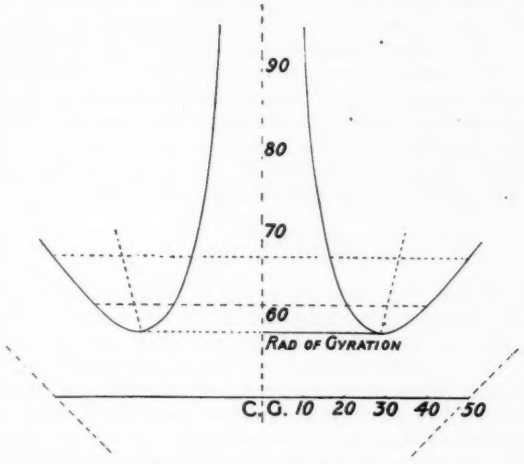
$$2x(mx+Mh)=I+i+mx^2,$$

$$i.e.\ \text{where } x=-\frac{Mh}{m}\pm\sqrt{\left(\frac{Mh}{m}\right)^2+\frac{I+i}{m}};$$

when m is in either of these positions, the corresponding simple pendulum is of length $2x$.

4. A common laboratory experiment on the compound pendulum is to swing a bar from a knife-edge whose position can be varied, and then to plot a curve shewing how the time of swing depends on where the knife-edge is, but (so far as I have seen) it is not customary to point out all that may be learned from such an experiment. Let us take x now to denote the height of the knife-edge above the C.G., which is regarded as origin; then the length of the simple equivalent pendulum is $\frac{I+Mx^2}{Mx}$, so that the graph in this case is $y=\frac{I+Mx^2}{Mx}$; the asymptotes are $x=0$ and $y=x$, of which the former corresponds to infinite period with the knife-edge at the C.G. If the knife-edge be moved still further, the bar hangs with the other end up, and we realise the other branch of the

hyperbola; the two branches may very well be plotted on the same side of the x-axis, and they will then be symmetrical about the y-axis. The minimum value of y will be twice the radius of gyration about the C.G., which is $\sqrt{\frac{I}{M}}$; and this corresponds to positions of the knife-edge at $x = \pm\sqrt{\frac{I}{M}}$. A line drawn across the graph parallel to the x-axis at any altitude exceeding $2\sqrt{\frac{I}{M}}$ will of course give four intersections, symmetrically placed with respect to the y-axis; and the distance between an asymmetric pair will be equal to the ordinate of the four points,



Uniform bar, a metre long.

since it will be the length of the simple pendulum equivalent to the compound pendulum swung from the four corresponding points. If the altitude of the line drawn exceeds $2\sqrt{\frac{I}{M}}$ only slightly, the four intersections will nearly coincide, two and two; and the length of the equivalent simple pendulum might then easily be misjudged owing to the selection of a symmetric instead of an asymmetric pair. This illustrates the point mentioned in Gray's *Text-book of Physics*, Vol. I., where the reader is cautioned against selecting two points equidistant from the C.G.; and it is stated that the outrageously wrong value obtained for g will usually reveal the nature of the mistake. It

might be added that with a pendulum reasonably designed such a mistake cannot fail to be revealed; for, as will presently be seen in another way, the very essence of proper adjustment is that the period should be such as to give four well separated intersections on the graph, or (in other words) to place the C.G. at very unequal distances from the centre of suspension and the centre of oscillation. (See § 9.)

5. Another matter worth mentioning in this connection is that as the graph is a conic, the locus of parallel chords is a straight line. Therefore, when the curve representing the observations has been plotted, a series of lines parallel to the x-axis should be drawn across it; if we then confine our attention to one branch at a time, the chords should be bisected and the middle points used in order to construct a "diameter" of the conic; this will intersect the curve at a point corresponding to the shortest period. The same thing having been done with the other branch, the distance between the two minima should be ascertained; the square of half this distance multiplied by the mass of the pendulum gives its moment of inertia about its C.G.

6. The simplicity of these results is often masked by plotting as ordinate the period, instead of the length of the equivalent simple pendulum; the graph has then the equation

$$\frac{y}{2\pi} = +\sqrt{\frac{Mx^2+I}{Mgx}},$$

a cubic whose only asymptote not entirely at infinity is $x=0$. It has no points below the x-axis or on the negative side of the y-axis, and in fact when the knife-edge is taken to the other side of the C.G. the equation requires a change of sign in the term Mgx; it then gives a new curve which is the reflection in the y-axis of the previous one, and corresponds to cases in which the pendulum hangs the other way up. It will be noticed that although in general a straight line cuts a cubic in three points, in this case a line parallel to the x-axis will only cut it in two; yet imaginary points can only come in in pairs. Of course the explanation is that the third point has gone away to infinity. Had there been three intersections at a finite distance, there must have been three points on the same side of the C.G. giving the same period; whereas we know there are only two. Similarly a line parallel to the y-axis meets the curve in one point at infinity, and the y-axis itself meets it in three points at infinity. The experiment is often carried out with a uniform bar drilled with holes along its length; each of these in succession is used as a means of supporting the bar upon a knife-edge. This method is preferable to attaching a knife-edge to the bar and sliding it to different positions. With a uniform bar the cubic character of the curve is not noticeable,

for the point of inflexion lies beyond the end of the bar; viz., at a distance from the centre equal to $\sqrt{1+\frac{2}{3}\sqrt{3}}$ times the half-length. However, by loading the bar at the centre the point of inflexion may be brought nearer; *e.g.* if at the C.G. of the bar there be attached a small mass having thrice the mass of the bar, then the point of inflexion is at only half the above distance.

If the experiment is carried out by means of a sliding knife-edge clamped to the bar, the only effect on the expression for the simple equivalent pendulum is to increase the numerator by one constant and the denominator by another, and the character of the graph is unchanged. A. O. ALLEN.

REVIEWS.

Volume and Surface Integrals used in Physics. By J. G. LEATHEM, M.A. (Cambridge University Press, 1905.) Price 2*s.* 6*d.* net.

This is No. 1 of a series of "Tracts" in the course of appearance dealing with mathematics and mathematical physics. So far as we can see, the object of these tracts is to deal with portions of mathematical theory too small to form a book, and either too large to form a paper or of too general application to remain buried in transactions until some specialist unearths them. If this view is correct, the useful purpose served by such tracts might almost justify their being multiplied indefinitely; indeed the "Tract" method of publication has many advantages over the "Transactions" method. The volume and surface integrals discussed in the book have special reference to quantities satisfying the partial differential equations of the first and second degrees occurring in physics. In the introduction the author discusses at some length the applicability of infinitesimal analysis to the properties of bodies whose ultimate structure is molecular, and suggests the term "physical smallness" as designating the order of magnitude of the elements involved in the analysis of physical bodies. Mr. Leathem's "physically small element" is thus the same as the "differential element" of the present reviewer. G. H. BRYAN.

Lectures on the Theory of Functions of Real Variables. By JAMES PIERPONT, Professor of Mathematics in Yale University. Volume I., xii + 560 pages. (Ginn & Co., Boston, [1905].)

In its historical development, the theory of functions of real variables is not always separable from the theory of analytic functions of complex variables. For example, Weierstrass' continuous function with no derivative was discovered by considering the real part of a certain analytic function on its circle of convergence, which is also a line of essential singularities; further, it casts, I think, some light on the theory of functions of a real variable to reflect that the first systematic treatment of this theory was made (by Hankel in 1870) long after the theory of analytic functions, as given by Cauchy and Riemann, had come to be widely known, and that it was made with the express

object of proving * that the limitation of the function-concept implied in the term "analytic" was a wise and, indeed, a necessary one. Lastly, that revision of fundamental principles, including the purely arithmetical introduction of irrational numbers, originated in the need for proofs of some vital theorems in the theory of analytic functions. To convince ourselves of this, we need only remember Weierstrass' avowed object that (at any rate, at first) he regarded the study of Abelian functions as his chief end in mathematics, and then glance at the pre-Weierstrassian "Théorie des fonctions doublement périodiques" of Briot and Bouquet, where, in No. 35, we can see the great importance for the theory of functions of Weierstrass' theorem that an infinite aggregate of points has at least one point of condensation, and, from No. 37, the importance of his distinction between upper limit and maximum, and his proof that a continuous function attains its maximum.†

While, then, Weierstrass' theory ‡ of irrationals, and its more amenable child, which is known as Cantor's theory, presupposes some acquaintance with the theory of analytic functions in order that we may fully realise their importance at the outset, Dedekind's theory, which arose from his attempts to deal with the proof of the existence of a limit in a purely arithmetical manner, would appear to be more suitable for beginners, and this view has found support in works of Pasch and J. Tannery. I am here doubting the advisability of any mode of exposition which leads to the remark of Prof. Pierpont (pp. 4-5): "It is too early to make the reader see the necessity of this step. . . ." If it is ever advisable to teach mathematics in this way, it is by introducing the student by a path which can never be used for the discovery of any important truths. It is possible, no doubt, to support such methods by the argument of the necessity of having a firm logical basis. But such arguments are never consistently applied. We do not abolish Euclid because of Mr. Russell's just strictures on its logic, and we are right; subtler logic is a result, not a prerequisite, of education. And I cannot but think that it is partly owing to unhistorical teaching that one so frequently sees the pitiable spectacle of a young man who has taken a high degree, and is therefore urged to undertake some mathematical research for a prize or a fellowship, not knowing what to do or how to set about finding anything to do.

Historically, the exact analysis of the number-concept comes late, and this analysis is perhaps the most difficult branch of mathematics. If an arithmetical theory of irrationals *is* to be put at the beginning of a book on function-theory, Dedekind's is preferable. These, in substance, are the criticisms which Prof. Pierpont's work suggests; and the criticisms apply only to the first few chapters of this otherwise admirable volume.

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* It succeeded in making this thesis plausible.

† Pages 46 and 48 of Fischer's German translation (1862) of Briot and Bouquet. Cf. pp. 47-48, 49 for a slight description of the ("essential," in Weierstrassian terms) singularity of $e^{\frac{1}{z}}$ at $z=0$.

‡ The same probably applies to Méray's theory, since Méray was, like Weierstrass, occupied with *analytic* functions.

A few remarks will show how easily mistakes creep into treatments of the number-concept. Prof. Pierpont postulates the theory of the positive integers 1, 2, . . ., and (p. 5) defines a fraction as a pair of integers (a, b), which has certain properties. He then (p. 6) identifies $(a, 1)$ with a. No integer can, under any circumstances, be identified with a pair of integers. In the same way '*positive* fractions' (pp. 12 *seq.*) are a different, though analogous, system to the fractions previously defined.

Rational numbers are classes of integers, as Prof. Pierpont says, and to most of us irrationals are classes of rationals (a definition which accords with, *e.g.*, Weierstrass', Lüroth's, and Russell's). But Prof. Pierpont merely says (p. 36): "Every regular sequence *defines* a number," but does not tell us *how*. He does not follow Heine's lead, and say it is a 'sign' (a proceeding which would make of arithmetic a trivial game), though a 'variable' is a 'symbol,' according to him (p. 118). When we speak of a 'variable,' we only mean a statement about *any* or *some* member of a certain class (of real or complex numbers usually).

From this point on all is excellent; the book is elementary, and is intended for those with a somewhat extensive, but uncritical knowledge of the calculus. It contains a rigorous treatment of continuity, differentiation, implicit functions, indeterminate forms, extremes, and single and multiple integrals (proper and improper). It does not deal with such advanced subjects as the condensation of singularities or continuous functions with no differential quotient anywhere.

Its chief excellencies seem to be: (1) The introduction of various current inaccurate statements as examples in criticism (pp. 231, 238, 240, 247, 251, 308, 321, 327, 371, etc.);* (2) analytical representations of some apparently 'lawless' functions. Thus

$$F(x) = f(x) + (g(x) - f(x)) \lim_{n=\infty} sgn(\sin^2 n!\,\pi x)$$

is $f(x)$ at rational, $g(x)$ at irrational points (pp. 202-205); (3) examples of continuous functions lacking differential quotients at isolated points, —functions often ignored by teachers (pp. 225-229); (4) the careful distinction between "any" and "every," which is often neglected.†

As a further instance of care, take Rolle's theorem (p. 246). It is necessary and sufficient that $f(x)$ should be continuous for the closed interval $a \leqq x \leqq b$, and $f'(x)$ should exist for every x such that $a < x < b$. If the aggregate for which $f(x)$ is continuous is not closed, the proof that $f(x)$ attains its extremes fails; and, on the other hand, the theorem holds even if there is (*e.g.*) no right-derivative at $x = a$. These facts are neglected in even such a book as Harnack's, which is usually considered by English students as almost unnecessarily rigorous.

The chapters on integration seem rather more advanced. On p. 356 (cf. p. 530) we have Jordan's conception of an integral defined

*Cauchy's definition of an irrational as a limit of rationals might be criticised when dealing with irrationals and the existence of limits.

Another suggestion I should like to make is that the author should illustrate that an always finite function need not be limited.

†Cf. Russell's *Principles of Mathematics*, Vol. I., 1903, p. 327 note. 'Every' is often used for 'any' in mathematical books of all nationalities.

over any measurable aggregate, while we have no mention (even afterwards) of say, continuous functions defined for any perfect aggregate, or of the four derivatives of a function.

The work is beautifully printed, and the only misprint I have seen is on p. 247, l. 15, where $f'(x)$ is put for $f(x)$.*

PHILIP E. B. JOURDAIN.

The Integration of Functions of a Single Variable. By G. H. HARDY. (No. 2 of the Cambridge Tracts on Mathematics and Mathematical Physics.)

This tract contains a number of striking and important theorems on indefinite integrals, which are due to Liouville very largely; as an example we may note (see p. 35) that $\log x = \int dx/x$ is *proved* not to be an algebraical function of x, a point which is not established in any of the ordinary books. Indications are given (p. 48) of the proof that the logarithm-integral $\int dx/\log x$, and the sine- and cosine-integrals $\int(\sin x/x)dx$, $\int(\cos x/x)dx$ are really new transcendents, not expressible in finite terms by any elementary functions.

As an example we may note that (p. 15) the integral

$$\int \frac{(ax^2+2\beta x+\gamma)dx}{(ax^2+2bx+c)^2}$$

is a rational function of x only if $a\gamma+c\alpha-2\beta b=0$, and is then equal to $-(\alpha x+\beta)/a(ax^2+2bx+c)$. And again (p. 33) that

$$\int \frac{dx}{(x-p)(ax^2+2bx+c)^{\frac{1}{2}}}$$

can only be algebraical if $ap^2+2bp+c=0$.

Nearly half the tract is devoted to the integrals of algebraical functions, with reference to which the fundamental theorem is due to Laplace: "*l'intégrale d'une fonction différentielle ne peut contenir d'autres quantités radicaux que celles qui entrent dans cette fonction.*" It is also explained (p. 36) that exponentials cannot occur in such integrals; and logarithms only in the form $A\log[\phi(x)]$, where A is a constant and $\phi(x)$ is an algebraical function containing only such irrationals as are present in the integrand. But as a contrast to the theory it may be noted that the method (Greenhill's) given on p. 24 for the integration of such an integral as

$$\int \frac{(x+2)dx}{X_1\sqrt{X}}, \quad X_1=(x+1)^2+2(x+2)^2, \quad X=(x+1)^2+(x+2)^2$$

would apparently express the integral in a form containing $\sqrt{X_1}$ as well as $\sqrt{X}$. However, by writing $\xi=(x+1)/\sqrt{X}$, it is not difficult to see that the integral is really a multiple of

$$\log\left[\frac{\sqrt{(2X)}-(x+1)}{\sqrt{(2X)}+(x+1)}\right].$$

* On p. 366. l. 20 (cf. pp. 206, 216), it would be better to put *lower limit* for *minimum* (which generally means an attained lower limit).

It seems a pity that space could not be spared to give an outline of the practical methods for integrals such as

$$\int \frac{dx}{X_1^{\frac{3}{2}} X^{\frac{1}{2}}}, \qquad X = x^2 - x + 1, \quad X_1 = x^2 + x + 1.$$

The ordinary description given in English books expects $(x^2+x+1)^{-3}$ to be first resolved into partial fractions, and then a formula of reduction has to be applied. Now a formula of reduction is extremely useful in certain special *definite* integrals, but is very inconvenient for *indefinite* integration. Undoubtedly the best method is to assume

$$\frac{1}{X_1^{\frac{3}{2}} X^{\frac{1}{2}}} = \frac{d}{dx}[(Ax^3 + Bx^2 + Cx + D)X^{\frac{1}{2}} X_1^{-2}] + (Ex + F)X^{-\frac{1}{2}} X_1^{-1},$$

and find $A, \ldots, F$, so as to make this an identity, since the calculation involves little more labour than the determination of the partial fractions, and no reduction formula is required. T. J. I'A. B

Integralrechnung. By W. FRANZ MEYER. (No. 11 of the *Sammlung Schubert.*) (Leipzig, G. J. Göschen.)

In many respects this book differs from the customary texts on Integral Calculus: as an example, the systematic discussion of types of integrals does not appear until p. 280, and a large number of special areas, volumes, etc., have been evaluated previously by comparatively elementary methods.

Some care is taken in the discussion of the formulae for the arc of a curve and for the area of a surface of revolution (§§ 11 and 13); in this respect it is pleasant to see an improvement on the crude methods adopted in most English books. In discussing the volume of an ellipsoidal or hyperboloidal slice, however (§ 10), an application of the prismoidal formulae,*

$$\tfrac{1}{6}h(A + B + 4M),$$

would have been simpler.

An interesting innovation in a Calculus is the large number of "Umkehrungen," in which curves are determined by means of some geometrical property; for example (§ 20) the curves in which (i) ρ/n, (ii) ρ/n^3, (iii) ρn are constant, where ρ is the radius of curvature and n is the normal. But it seems strange to an English reader that the formulae for the radius of curvature and allied results are established in an Integral (instead of a Differential) Calculus; and that some 20 pages are devoted to the curvature of twisted curves and surfaces.

In §§ 35, 36 are given some theorems on the elliptic integrals and elliptic functions; the addition-theorem is obtained from the integrals (using Darboux's method) in the form:

$$\text{If} \qquad u = \int_0^x X^{-1} dx, \text{ and } x = f(u), \text{ where } X^2 = ax^4 + 2bx^2 + c,$$

$$\text{then} \qquad f(u+v) = c^{\frac{1}{2}}[f(u)f'(v) + f(v)f'(u)]/[c - a\{f(u)f(v)\}^2].$$

* Here h is the thickness, A and B the areas of the two parallel end faces, M the area of the central section parallel to the faces. The formula applies to any solid in which the area of the section is a quadratic (or cubic) function of x, the distance of the section from an end. For an ellipsoid the result is $\frac{1}{6}Ah + \frac{1}{8}A_0h^3/p^2$, where A_0 is the area of the parallel central section and p is the perpendicular on the parallel tangent-plane.

This, of course, includes the addition-theorems for the hyperbolic and trigonometrical functions as special cases.

In § 38 some of the familiar power series are derived by means of the Integral Calculus; and this is a very good method for many of the series. Thus the series for arc $\sin x$ is obtained far more easily from $\int_0^x (1-t^2)^{-\frac{1}{2}} dt$ than by applying Taylor's theorem. The book closes with a section on double integrals and their applications.

It can hardly be doubted that most teachers of the Calculus will find many points of interest and means of improving their teaching in this book. T. J. I'A. B.

Theorie der ebenen algebraischen Kurven höherer Ordnung. By HEINRICH WIELEITNER. (No. 43 of the *Sammlung Schubert.*) (Leipzig, G. J. Göschen.)

This text-book covers to some extent the same ground as Salmon's *Higher Plane Curves*, the harder portions being omitted. But several points are considered more fully than by Salmon.

Perhaps the most striking contrast between the two books may be found in the profusion of diagrams given by Wieleitner; nearly 50 of these have been drawn accurately to scale, from the corresponding equations. In particular, the diagram of the parallel curve to an ellipse (p. 288, fig. 73) is extremely instructive, quite apart from its beautiful drawing.

Amongst other matters not given by Salmon we may refer to the constant use of Newton's diagram for the discussion of singularities on a curve; to the resolution of higher singularities into nodes, cusps, inflexions and bitangents; and to Klein's theorem on *real* singularities. Frequent use of Plücker's "Auflösung" of a node is made; thus (p. 92) from the line and circle $y(x^2+y^2-1)=0$, we can get various families of cubics, such as

$$y(x^2+y^2-1)+\lambda=0, \quad y(x^2+y^2-1)+\lambda x=0, \quad y(x^2+y^2-1)+\lambda x^2=0.$$

Further, when λ is *small*, these cubics differ but little from the line and circle, which enables us to realize their shape very easily.

The book seems to be a very useful introduction to the theory of Higher Plane Curves. T. J. I'A. B.

Mehrdimensionale Geometrie. (II. Teil.) By P. H. SCHOUTE. (No. 36 of the *Sammlung Schubert.*) (Leipzig, G. J. Göschen.)

According to the preface, the author was forced, for lack of space, to compress his MS. into about three-fourths of the original number of pages; and this accounts for the occasionally rather too compact form of the argument.

The subject-matter corresponds roughly to the work given, for ordinary space, in such books as Nixon's *Geometry in Space.* The book contains extensions of Euler's theorem, of the theory of regular polyhedra, of spherical geometry, and so on; the work is well illustrated by numerous carefully drawn figures. We may mention specially the figures, for ordinary space, of the regular and semi-regular solids.

T. J. I'A. B.

The Theory of Determinants in the Historical Order of their Development. Second edition. By T. MUIR, LL.D., F.R.S. 1906. (Macmillan.)

The quality of the research undertaken by Dr. Muir in a subject he has made peculiarly his own, is too well known to every mathematician who is interested in the historical development of the theory of determinants to render it necessary for us to do more than to call the attention of our readers to the completed volume which now lies before us. *Part I.* was published sixteen years ago, and no doubt the accomplished author looked forward to the appearance of Part II. within a reasonable period after that of its predecessor. But the proposals of man are often disposed of in an unexpected manner, and the pressure of official work in a new sphere of interests and utility proved the truth of the adage. As time permitted, the lists which appeared in the *Quarterly Journal* were added to, and from the titles of the various papers collated by the author, 1740 in number, we see he has brought his labours up to the year 1900. The present volume is said to be a second edition. This is true only of the first section, covering 288 pages. .Such intercalations are here made as were found necessary from discoveries made since the first volume appeared. The total amount of matter added to what was to be found in the old edition is some fifteen pages. The remaining 200 pages are what would have been Vol. II., but it was considered advisable to place the whole within one moderately-sized volume, and this is what has been done. Let us say at once that in one respect the new volume is easier to read than its predecessor—it is in larger type, and where the reader who has passed the bounds of middle age can read large type he prefers to do so as a rule. Referring to the pages of the original volume we may note a few of the additions and alterations. By a strange slip in the first edition the memoir by Garnier (1814) on p. 215, was out of its chronological place. This has been corrected, and it is now on pp. 135-6. A memoir by Cauchy (1829), dealing with the equation by means of which the secular irregularities of the movements of the planets are determined, was overlooked. Strictly speaking, the determinants in this memoir are axisymmetric, and would, therefore, fall into Part II., but as the proof given for one of the theorems in the memoir is applicable to determinants of every type, it is now included here, and placed between the memoirs of Reiss and Jacobi on p. 184 of the first edition. The one-page summary of Jacobi's memoir, just mentioned, is now replaced by five-and-a-half pages. The summary is much more detailed, and includes a proposition in which is given the generating function of the reciprocal of a determinant. At the old page, 201, after the Latin quotation, we find three pages inserted, giving Jacobi's method of proving a certain pair of theorems. This is followed by a statement of a theorem arrived at in 1834 by Jacobi, which is also new. A short summary of a paper by Molins, 1839, is introduced just before the analysis of Sylvester's article of 1840 in the *Phil. Mag.*, vol. xvi. In the note to p. 237, first edition, "a term of the cyclic-species" is altered to "a term whose index-cycle scheme." On p. 259 there is a small addition. On p. 288 appears a page of tables similar to that opposite p. 132. This rams home the facts of the second period as has already been done for the first. We now reach the special determinants which make up the subject matter of the new Part II. The various chapters deal with axisymmetric and skew determinants, the miscellaneous determinants associated with the names of Wronski, Scherk, Schweins, Jacobi, and Sylvester, and with Alternants, Jacobians, and Orthogonants. A short "Retrospect on Special Forms," with indexes, brings to a close a piece of work of which English mathematicians need not be ashamed.

An Elementary Treatise on Pure Geometry, with numerous examples; by J. W. RUSSELL. New and Revised Edition, 1905. (Clarendon Press.)

The only fault that can be found with Mr. Russell's admirable Geometry is that it is far too full for ordinary school purposes. As a book of reference it is invaluable, for all the properties that a teacher or student will ever be likely to require are here to be found. In the new edition we find a model at the end of the book which will prove of much value in the study of the chapter on projection to those whose intuitive faculty is not highly developed. The questions have been

re-arranged, those that were redundant have disappeared, all are now in larger type, and the problems are relegated to the ends of the chapters. Many a weary hunt will be saved to future readers by the adequate index. The author tells us in his preface that the method of one to one correspondence has been introduced wherever possible, but only as an alternative method. He thinks it, "though elegant and powerful, a dangerous method, on account of the difficulty of seeing whether a given construction is rational." In the proofs of the theorems more use has been made of projection, and correlative theorems have been proved by reciprocation. While we admit that all the changes in the new and revised edition have been for the better, we feel sure that we are echoing the feelings of many teachers when we regret that the time thus spent was not rather devoted to the production of a text-book which can be used in schools. For such a book, written by one who has such a mastery of the subject, we are convinced there would be a large demand.

MATHEMATICAL NOTES.

191. [**v. a.**]. The "appeal" in the last number of the *Gazette* has encouraged me to send a miscellaneous collection of ideas, some suggested by the contents of the number, some of slower growth.

(1) *Higher Trigonometry.* It is encouraging to have Mr. G. H. Hardy's sanction for some heretical notions that many schoolmasters have for some time entertained secretly. As to the proof of $\cos x+i\sin x=\exp(ix)$ I have used, to my own satisfaction, the proof Mr. Hardy recommends: I have been in the habit of giving it to ordinary scholarship candidates, and they did not find it difficult. I should be glad to know if Mr. Hardy would pass this form (which is borrowed from the official program of L'École des Arts et de la Commerce).

Let $$\cos x+i\sin x=\exp(y). \quad \text{.....(i)}$$

Differentiating, $$-\sin x+i\cos x=\exp(y)\frac{dy}{dx}; \quad \text{.....(ii)}$$

$$\therefore \frac{dy}{dx}=\frac{-\sin x+i\cos x}{\cos x+i\sin x}=i; \quad \text{.....(iii)}$$

$$\therefore y=ix+c; \quad \text{.....(iv)}$$

$$\therefore \cos x+i\sin x=\exp(ix+c)$$
$$=\exp(ix).\exp c. \quad \text{.....(v)}$$

Put $x=0$; then $1=\exp c$;

$$\therefore \cos x+i\sin x=\exp(ix). \quad \text{.....(vi)}$$

(i) assumes the existence of a root to a transcendental equation for y; (ii) arises from the term by term differentiation of a series. Am I right in supposing that it is unnecessary to enter upon a discussion of contour integration before taking step (iv)?

(2) *Contracted Multiplication and Division.* This used to be one of the main planks of the mathematical reformer's platform. I suppose that the importance attached to these rules was a sign of the reaction against the unpractical, vulgar-fractions-and-farthings type of arithmetic. Certainly, for my own part, I used to swear by these rules; I was brought up on them, and for several years taught them with enthusiasm. They were hard work, for both teacher and taught: but both were kept going by the thought of the vast saving in labour to be effected by the use of the rules. It is true that I never actually *used* them myself, as I preferred logarithms; and I never managed to make any boy use them except to win marks. Before a boy could use the rules, he had to revise them in class for a week or so; and the better boys said that to use the rules was such a strain upon their minds that they would rather work without contracting if they had no log

books. However, we pegged away. One day I wrote to several engineers to ask their opinion as to the value of the rules in real life. One said that he used the slide-rule himself; another said that, to answer my questions, he had enquired what the rules were, and he thought they must be capital things. However, on the whole I came to the conclusion that they were but toys; that it was much easier to teach logarithms straight away; and that a great quantity of the best teaching power is being squandered in teaching rules which no one would ever think of applying except under penalties. Accordingly, I cut them out of the curriculum for which I was responsible: but made careful provision for teaching the boys in other ways such notions as significant figures, degree of approximation, etc.

I think that this is a topic which might well be discussed in your pages.

(3) *Division.* I wonder if other teachers have noticed the extreme helplessness of boys of thirteen in dealing with a division sum. The difficulty is to know which number is to be the divisor. There is, of course, a preliminary difficulty—whether to divide or subtract. "If 40 sacks of coal weigh 2240 lbs., what does 1 sack weigh?" It seems to be entirely a matter of chance whether the answer is given as $\frac{2240}{40}$ or $\frac{40}{2240}$ lbs. The only principle grasped firmly is—divide by the smaller number, for choice.

In saying that it is a matter of chance "which way" the boy divides, I am not going far enough. I seriously believe that he is more likely to divide the wrong way; this belief is based on recent experience.

This may seem a small matter, but it tends to vitiate the whole of a boy's arithmetic. In searching for the causes of this curious difficulty, I find that the following suggest themselves:—

(A) Division is, and must be, taught as a mechanical process. But is it also taught, as it should be, in connection with the meaning? In other words, a boy must know how to work out the quotient to $516 \div 3$; and he always does know this. But I am beginning to think that he often forms no concrete idea of the kind of case to which division is applicable.

The remedy for this is practical work at the earliest stages—kindergarten work; cutting up paper into so many parts, dividing shot into so many parcels, etc. (partition); and also the other kind of division—how many sets of 3 counters in a heap of 60, how many times will the yard-measure fit into the length of the room and how much is left (quotition)?

(B) The other cause of the difficulty, to my mind, is that vile phrase "divided into"—"twelve divided into three" (meaning $3 \div 12$). It will be noticed that by adding the word "parts" we reverse the operation "twelve divided into three parts" gives $12 \div 3$. Is it surprising that a boy accustomed to "divide into" is a bit foggy as to which way the division is to go in a concrete case?

CHARLES GODFREY.

192. [A. 1. b.] *Note on the Power Inequality.*

The inequality in question, which is the fundamental one in *Chrystal*, may be stated and proved as follows:

If x and y be positive and $x > y$, and m be any rational number not equal to 1 *or* 0, *then,* $mx^{m-1} \gtrless \frac{x^m - y^m}{x - y} \gtrless my^{m-1}$, *according as* $m(m-1) \gtrless 0$.

Proof.—From the identity $\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \ldots + y^{n-1}$, we get at once,

$$nx^{n-1} > \frac{x^n - y^n}{x - y} > ny^{n-1},$$ n being any integer > 1. .....(1)

The above inequality can be written, as is readily verified,

$$\frac{n}{n-1}x > \frac{x^n - y^n}{x^{n-1} - y^{n-1}} > \frac{n}{n-1}y.$$

Multiplying a number of such inequalities together we get, if p and q be positive integers, $p > q$,

$$\frac{p}{q}x^{p-q} > \frac{x^p - y^p}{x^q - y^q} > \frac{p}{q}y^{p-q}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

Whence, writing $x^{\frac{1}{q}}$, $y^{\frac{1}{q}}$ for x, y and m for p/q,

$$mx^{m-1} > \frac{x^m - y^m}{x - y} > my^{m-1}, \text{ if } m > 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (a), considering the reciprocals of the terms, we get,

$$\frac{q}{p}x^{q-p} < \frac{x^q - y^q}{x^p - y^p} < \frac{q}{p}y^{q-p}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

Whence, writing $x^{\frac{1}{p}}$, $y^{\frac{1}{p}}$ for x, y and m for q/p,

$$mx^{m-1} < \frac{x^m - y^m}{x - y} < my^{m-1}, \text{ if } 1 > m > 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Again from the identity

$$\frac{\left(\frac{1}{x}\right)^n - \left(\frac{1}{y}\right)^n}{x - y} = -\left(\frac{1}{x^n y} + \frac{1}{x^{n-1}y^2} + \ldots + \frac{1}{xy^n}\right),$$

by considering the effect of increasing y to x, and then the effect of decreasing x to y, on the right-hand side, we get,

$$-nx^{-n-1} > \frac{x^{-n} - y^{-n}}{x - y} > -ny^{-n-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Similarly from the identity

$$\frac{\left(\frac{1}{x}\right)^p - \left(\frac{1}{y}\right)^p}{x^q - y^q} = -\frac{\frac{1}{x^p y} + \frac{1}{x^{p-1}y^2} + \ldots + \frac{1}{xy^p}}{x^{q-1} + x^{q-2}y + \ldots + y^{q-1}},$$

we get,

$$-\frac{p}{q}x^{-p-q} > \frac{x^{-p} - y^{-p}}{x^q - y^q} > -\frac{p}{q}y^{-p-q}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c)$$

Whence, writing $x^{\frac{1}{q}}$, $y^{\frac{1}{q}}$ for x, y, and m for $-p/q$, we get,

$$mx^{m-1} > \frac{x^m - y^m}{x - y} > my^{m-1}, \text{ } m \text{ being any negative quantity.} \quad \ldots\ldots\ldots\ldots (5)$$

Results (1), (2), (3), (4) and (5) prove the theorem.

Gooty, India, 8.4.06. V. RAMASWAMI AIYAR.

193. [C. 2. h.] The following Exercise has the advantage of obtaining two well-known integrals simultaneously:

$$I = \int_0^\infty \frac{x^{2m}\,dx}{1 + x^{2n}}, \quad m < n,$$

$$I' = \int_0^\infty \frac{x^{2m}\,dx}{1 - x^{2n}}, \quad m < n.$$

Let ODE be a sector of infinite radius R and angle $\frac{\pi}{2n}$; then the function $\frac{x^{2m}}{1 - x^{2n}}$ has one pole $P(x = 1)$ on OD and no pole inside sector.

Let ABC be a semi-circle centre P and infinitely small radius r.

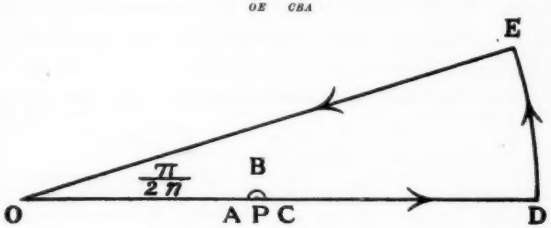
Integrating $\frac{x^{2m}}{1 - x^{2n}}$ over the contour excluding P, we have

$$\int_{OA} + \int_{ABC} + \int_{CD} + \int_{DE} + \int_{EO} = 0.$$

Now the 1st and 3rd make up (for $r=0$) the "principal value" of I'.

Since $\underset{x=\infty}{\mathrm{Lt}}\, x \cdot \frac{x^{2m}}{x^{2n}}$ is 0, the 4th term $=0$.

Hence
$$I' - \int_{OE} = \int_{CBA}.$$



Now
$$\int_{OE} = \int_0^R \frac{(te^{\pi\iota/2n})^{2m} e^{\pi\iota/2n} dt}{1+t^{2n}} \quad \because\ e^{\pi\iota} = -1$$
$$= e^{(2m+1)\pi\iota/2n} I,$$

and
$$\int_{CBA} = \underset{r=0}{\mathrm{Lt}} \int_0^\pi \frac{(1+re^{\iota\phi})^{2m} re^{\iota\phi}\iota d\phi}{1-(1+re^{\iota\phi})^{2n}} = \int_0^\pi \frac{\iota d\phi}{-2n} = -\frac{\pi\iota}{2n};$$

$$\therefore\ I' - \{\cos(2m+1)\pi/2n + \iota \sin(2m+1)\pi/2n\} I = -\frac{\pi\iota}{2n};$$

$$\therefore\ I = \frac{\pi}{2n} \Big/ \sin(2m+1)\pi/2n,$$

and
$$I' = \cos(2m+1)\pi/2n \,.\, I$$
$$= \frac{\pi}{2n} \cot(2m+1)\pi/2n.$$

R. W. GENESE.

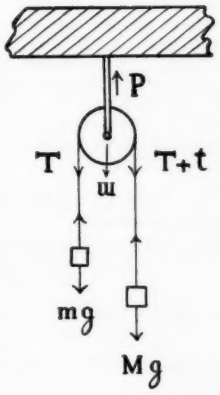
194. [R. 9; K. 9. b.] 1. *Inexactitudes in the treatment of simple exercises on the motion of unequal masses suspended over a pulley.*

(*a*) Some text-books fail to indicate clearly the idealisations involved in supposing the tension of the string to be the same throughout.

(*b*) When these several points are made clear, the reader—presumably a beginner—is left to imagine that, if two unequal masses could be suspended over a perfectly smooth pulley, this pulley would rotate.

It seems unfortunate that the part played by friction is not more clearly brought out. The ideal problem could be treated by supposing the string to pass over a smooth *peg* and by pointing out the fact that a close approximation is thus obtained to the actual motion which takes place when it passes round a suitably mounted pulley. The diagram for the "pulley" problem would then be somewhat like the accompanying figure.

[t negligible in considering the motion of M, m.]



2. *Construction of a regular pentagon.*

(*a*) The method based on Euclid IV. 10 is essentially unpractical.

(*b*) Simple constructions, easy to perform with accuracy, are given in Plant's *Geometrical Drawing*.

(*c*) If the latter are adopted, as seems advisable, what line of argument furnishes the easiest and most natural proofs?

Tentative answer.

(i) Consider a right-angled triangle ABC, right-angled at C, such that $\frac{BC}{CA}=\frac{\sqrt{5}-1}{4}$. Using the '$2A$' formulæ, prove that $\cos 4A=\sin A$, thus obtaining $\sin 18°$.

(ii) Employ this result in demonstrating the correctness of the constructions referred to. A. F. VAN DER HEYDEN.

195. [X. 1.] *Note on the number of feet in a metre.*

A high order of accuracy is secured in calculations by taking 1 metre = 39·37079 inches, this being the actual value of a metre to five places of decimals. The object of this note is to show that to this order of accuracy the very simple fraction $\frac{292}{89}$ of a foot may be taken as the equivalent of a metre.

The proof is simple. Expand 39·37079 into a continued fraction. The seventh convergent is $\frac{3504}{89}$ and the eighth quotient is 35; thus $\frac{3504}{89}$ differs from 39·37079 by less than $\frac{1}{35}(\frac{1}{89})^2$, *i.e.* by less than ·0000037. Thus to five places of decimals they are identical, and to this order $\frac{3504}{89}$ inches, *i.e.* $\frac{292}{89}$ feet, represents a metre. H. L. TRACHTENBERG.

196. [L_1. 3. a.] *Cartesian Coordinates—A new method of finding the equation of the axis of a parabola.*

Let the parabola be $ax^2+2hxy+by^2+2gx+2fy+c=0$. ....(1)

Then the equation of its axis is obtained by equating to zero the linear portion of the equation,

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}-\frac{(ax+hy+g)(hx+by+f)}{h}=0. \quad \text{.....(2)}$$

For had (1) been any central conic, (2) would have been the equation of its axes. But as the centre of a conic goes towards infinity, one of these axes tends to become the line at infinity. Thus in the case of a parabola one of the factors of (2) is a constant, *i.e.* the equation (2) reduces to a constant multiplied by the factor representing the finite axis. Thus the rule given above for obtaining the equation of the axis of any parabola is true. Since the highest terms of (2) have to vanish in the case of a parabola, we equate them to zero and get the condition that (1) should be a parabola. The vanishing of the three coefficients of x^2, xy, and y^2, give the same condition $h^2=ab$. H. L. TRACHTENBERG.

197. [L_1. 3. a.] *Trilinear Coordinates. The equation of the axis of the general parabola.*

The equation of the axes of any central conic

$$ua^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma a+2w'a\beta=0$$

is
$$\begin{vmatrix} 2bc(w'a'+v\beta'+u'\gamma)(v'a+u'\beta+w\gamma)-c^2(w'a+v\beta+u'\gamma)^2 \\ \qquad -b^2(v'a+u'\beta+w\gamma)^2, \quad 2bcu'-c^2v-b^2w, \quad 1 \\ 2ca(v'a+u'\beta+w\gamma)(ua+w'\beta+v'\gamma)-a^2(v'a+u'\beta+w\gamma)^2 \\ \qquad -c^2(ua+w'\beta+v'\gamma)^2, \quad 2cav'-a^2w-c^2u, \quad 1 \\ 2ab(ua+w'\beta+v'\gamma)(w'a+v\beta+u'\gamma)-b^2(ua+w'\beta+v'\gamma)^2 \\ \qquad -a^2(w'a+v\beta+u'\gamma)^2, \quad 2abw'-b^2u-a^2v, \quad 1 \end{vmatrix}=0.$$

As the conic tends to become a parabola, one of the axis tends to become the line at infinity. Thus if the first equation be that of a parabola, the determinant $\equiv(a\alpha+b\beta+c\gamma)(l\alpha+m\beta+n\gamma)$, where $l\alpha+m\beta+n\gamma=0$ is the equation of the axis. Thus $l\alpha$=the coefficient of α^2 in the determinant, and l is found. Similarly m and n are found. Thus the equation of the axis is

$$\Sigma\Big\{[(2bcw'v'-c^2w'-b^2v')\{(2cav'-a^2w-c^2u)-(2abw'-b^2u-a^2v)\}$$
$$+(2cav'u-a^2v'^2-c^2u^2)\{(2abw'-b^2u-a^2v)-(2bcu'-c^2v-b^2w)\}$$
$$+(2abuw'-b^2u^2-a^2w'^2)\{(2bcu'-c^2v-b^2w)-(2cav'-a^2w-c^2u)\}]\frac{a}{a}\Big\}=0,$$

i.e. $\Sigma[(cw'-bv')^2\{(2cav'-a^2w-c^2u)-(2abw'-b^2u-a^2v)\}$
$$+(av'-cu)^2\{(2abw'-b^2u-a^2v)-(2bcu'-c^2v-b^2w)\}$$
$$+(bu-aw')^2\{(2bcu'-c^2v-b^2w)-(2cav'-a^2w-c^2u)\}]\frac{\alpha}{a}=0.$$

H. L. TRACHTENBERG.

198. [L¹. 3. a.] *A new method of finding the equation of the axes of the general conic.*

$$\lambda(ax^2+2hxy+by^2+2gx+2fy+c)$$
$$=[(ax_0+hy_0+g)x+(hx_0+by_0+f)y+(gx_0+fy_0+c)]^2$$

represents the system of conics having double contact with

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

at the points where it is cut by the polar of x_0y_0. Now if x_0y_0 is on an axis, and in that case alone, one of the conics of this system will be a circle, for the tangents from such a point to the conic are equal. Let λ_1 be the value of λ for this circle. Then the conditions for a circle give

$$\lambda_1a-(ax_0+hy_0+g)^2=\lambda_1b-(hx_0+by_0+f)^2,$$
$$\lambda_1h-(ax_0+hy_0+g)(hx_0+by_0+f)=0.$$

Thus, eliminating λ_1 the equation of the axes is

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)(hx+by+f)}{h}.$$

H. L. TRACHTENBERG.

199. [L¹. 3. a.] *Trilinear Coordinates.—The equation of the axes of the general conic.*

$$\lambda(u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta)$$
$$=[\alpha(u\alpha'+w'\beta'+v'\gamma')+\beta(v'\alpha'+v\beta'+u'\gamma')+\gamma(v'\alpha'+u'\beta'+w\gamma')]^2$$

is the system of conics touching the conic

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$$

at the points where it is met by the polar of $\alpha'\beta'\gamma'$. If $\alpha'\beta'\gamma'$ is on an axis, and in that case alone, one of the conics of this system will be a circle. Let λ_1 be the corresponding value of λ. Then writing down the conditions for a circle, a, b, c being the sides of triangle of reference

$$2bc[\lambda_1u'-(w'\alpha'+v\beta'+u'\gamma')(v'\alpha'+u'\beta'+w\gamma')]$$
$$-c^2[\lambda_1v-(w'\alpha'+v\beta'+u'\gamma')^2]-b^2[\lambda_1w-(v'\alpha'+u'\beta'+w\gamma')^2]$$
$$=2ca[\lambda_1v'-(v'\alpha'+u'\beta'+w\gamma')(u\alpha'+w'\beta'+v'\gamma')]$$
$$-a^2[\lambda_1w-(v'\alpha'+u'\beta'+w\gamma')^2]-c^2[\lambda_1u-(u\alpha'+w'\beta'+v'\gamma')^2]$$
$$=2ab[\lambda_1w'-(u\alpha'+w'\beta'+v'\gamma')(w'\alpha'+v\beta'+u'\gamma')]$$
$$-b^2[\lambda_1u-(u\alpha'+w'\beta'+v'\gamma')^2]-a^2[\lambda_1v-(w'\alpha'+v\beta'+u'\gamma')^2].$$

Putting each of these three equal quantities $=k$, and eliminating λ_1 and k from the three linear equations thus formed, we have for the equation of the axes

$$\begin{vmatrix} 2bc(w'\alpha+v\beta+u'\gamma)(v'\alpha+u'\beta+w\gamma)-c^2(w'\alpha+v\beta+u'\gamma)^2 \\ \qquad -b^2(v'\alpha+u'\beta+w\gamma)^2, \quad 2bcu'-c^2v-b^2w, \quad 1 \\ 2ca(v'\alpha+u'\beta+w\gamma)(u\alpha+w'\beta+v'\gamma)-a^2(v'\alpha+u'\beta+w\gamma)^2 \\ \qquad -c^2(u\alpha+w'\beta+v'\gamma)^2, \quad 2cav'-a^2w-c^2u, \quad 1 \\ 2ab(u\alpha+w'\beta+v'\gamma)(w'\alpha+v\beta+u'\gamma)-b^2(u\alpha+w'\beta+v'\gamma) \\ \qquad -a^2(w'\alpha+v\beta+u'\gamma)^2, \quad 2abw'-b^2u-a^2v, \quad 1 \end{vmatrix} =0.$$

H. L. TRACHTENBERG.

200. [L². 2. e; 4. a.] *A new method of finding the condition that the general conicoid should be one of revolution and of finding the equation of its axis.*

$$\begin{aligned} &\{(ax_0+hy_0+gz_0+u)x+(hx_0+by_0+fz_0+v)y \\ &\qquad +(gx_0+fy_0+cz_0+w)z+(ux_0+vy_0+wz_0+d)\}^2 \\ &\qquad =\lambda[ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d] \end{aligned}$$

is the system of conicoids touching

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2fy+2wz+d=0$$

round the curve of section by the polar plane of $x_0y_0z_0$. If (1) the surface is one of revolution *and* (2) $x_0y_0z_0$ is on the axis of revolution, and under these conditions alone, one of the conicoids of this system will be a sphere, for all the tangents from $x_0y_0z_0$ to the conicoid are equal. Let λ_1 be the value of λ for this sphere. Then writing down the conditions for a sphere

$$\left.\begin{aligned} &(ax_0+hy_0+gz_0+u)^2-\lambda a=(hx_0+by_0+fz_0+v)^2-\lambda b=(gx_0+fy_0+cz_0+w)^2-\lambda c, \\ &(hx_0+by_0+fz_0+v)(gx_0+fy_0+cz_0+w)-\lambda f=0, \\ &(gx_0+fy_0+cz_0+w)(ax_0+hy_0+gz_0+u)-\lambda g=0, \\ &(ax_0+hy_0+gz_0+u)(hx_0+by_0+fz_0+v)-\lambda h=0. \end{aligned}\right\}$$

Thus, eliminating λ, the axis lies on all the following surfaces.

$$(\alpha)\ldots\ldots\ldots\ldots\left\{\begin{aligned} &\frac{(hx+by+fz+v)^2-(gx+fy+cz+w)^2}{b-c} \\ &\qquad =\frac{(gx+fy+cz+w)^2-(ax+hy+gz+u)^2}{c-a} \\ &\qquad =\frac{(ax+hy+gz+u)^2-(hx+by+fz+v)^2}{a-b} \\ &\qquad =\frac{(hx+by+fz+v)(gx+fy+cz+w)}{f} \\ &\qquad =\frac{(gx+fy+cz+w)(ax+hy+gz+u)}{g} \\ &\qquad =\frac{(ax+hy+gz+u)(hx+by+fz+v)}{h}. \end{aligned}\right.$$

Now three of these surfaces are the three pairs of planes

$$\begin{aligned} &(ax+hy+gz+u)\{h(gx+fy+cz+w)-g(hx+by+fz+v)\}=0. \\ &(hx+by+fz+v)\{f(ax+hy+gz+u)-h(gx+fy+cz+w)\}=0. \\ &(gx+fy+cz+w)\{g(hx+by+fz+v)-f(ax+hy+gz+u)\}=0. \end{aligned}$$

Call these $\phi_1 . F_1 = 0$, $\phi_2 . F_2 = 0$, $\phi_1 . F_3 = 0$. If we can find one line and one only, that is common to all three pairs, that line must be the axis. Now the planes $F_1 = 0$, $F_2 = 0$, $F_3 = 0$ have a common line for $F_1 + F_2 + F_3 \equiv 0$ and *in general* no other three planes, one from each pair, have a common line. Thus in general the axis is given by

$$(\beta) \ldots\ldots\ldots f(ax+hy+gz+u) = g(hx+by+fz+v) = h(gx+fy+cz+w).$$

Putting each of these $= K$ and solving the equations we have

$$\frac{x}{\begin{vmatrix} h & g & u-K/f \\ b & f & v-K/g \\ f & c & w-K/h \end{vmatrix}} = \frac{y}{\begin{vmatrix} g & a & u-K/f \\ f & h & v-K/g \\ c & g & w-K/h \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & h & u-K/f \\ h & b & v-K/g \\ g & f & w-K/h \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}}.$$

Thus the equation of the axis is

$$\frac{x-\frac{U}{D}}{A'/f+H'/g+G'/h} = \frac{y-\frac{V}{D}}{H'/f+B'/g+F'/h} = \frac{z-\frac{W}{D}}{G'/f+F'/g+C'/h}.*$$

Also using the fact (β) that

$$ax+hy+gz+u : hx+by+fz+v : gx+fy+cz+w = 1/f : 1/g : 1/h$$

we get from (α)

$$\frac{1/g^2-1/h^2}{b-c} = \frac{1/h^2-1/f^2}{c-a} = \frac{1/f^2-1/g^2}{a-b} = \frac{1}{fgh},$$

$$i.e.\ b-c = \frac{hf}{g} - \frac{fg}{h} \text{ and two similars,}$$

$$i.e.\ \frac{gh}{f} - a = \frac{hf}{g} - b = \frac{fg}{h} - c$$

are the conditions for a surface of revolution. H. L. TRACHTENBERG.

Prof. E. B. Elliott's note (*Gazette*, Oct. 1905, p. 236) seems fallacious. To show this, suppose that the convergent sequence $s_1, s_2, \ldots$, of rationals tend to *no rational limit*. Then, given any positive rational ϵ', there is a (one or more) rational s such that, for some n and every p, $|s - s_{n+p}| < \epsilon'$. But, as ϵ' is diminished to zero, we can find *no* rational s valid for *all* such ϵ's. Thus the conclusion of (1) fails. The fact is, that logically a definition of a real number (*i.e.*, a sequence $s_1, s_2, \ldots$, of rationals such that, where ϵ is any positive rational, there is an integer n, which depends on ϵ, such that, for every p, $|s_n - s_{n+p}| < \epsilon$) precedes the criterion for the existence of a limit, and that then the sufficiency of this criterion follows at once (cf. *Encykl. der Math. Wiss.*, I. A 3, §§ 5, 6, 12, 13); though geometrical ways of thinking obscured this logical fact, even with Cauchy (cf. my remarks in *Bibl. Math.*, 1905, p. 206), until the time of Weierstrass.

PHILIP E. B. JOURDAIN.

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* The dashed capitals are minors of the corresponding small letters in $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

The undashed capitals are minors of the corresponding small letters in $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$.

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